# Nambu brackets in fluid mechanics and magnetohydrodynamics

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## Introduction to Nambu Formalism

Here we propose a Nambu formalism for Boussinesq 3D (also 2D in the submitted paper) and Magnetohydrodynamics (MHD)
To define the Nambu formalism consider, following [1], a triplet of dynamical variables $\mathbf{r} = (x, y, z)$ that spans a 3D phase space. Then introduce two functions $H$ and $G$ of $(x, y, z)$, where $H$ is the Hamiltonian of the system and $G$ a Casimir. After, we have for an arbitrary function $F = F(x, y, z)$ :

$$\frac{dF}{dt} = \frac{\partial(F, H, G)}{\partial(x, y, z)} \equiv \nabla F \cdot (\nabla H \times \nabla G) = [F, H, G] \quad (1)$$

The term on the right side of equation (1) is called Nambu bracket (NB).
We call Nambu brackets of first kind (NB I) the NBs (1) or sums of NBs which are related to different triplets $(x_k, y_k, z_k)$, $k = 1, \ldots, N$, that involve the same conserved quantities $H$ and $G$ [1]:

$$\frac{dF}{dt} = \sum_k \frac{\partial(F, H, G)}{\partial(x_k, y_k, z_k)} = \sum_k [F, H, G]_k \quad (2)$$

The sums of NBs that involve different conserved or constitutive quantities $(H_i, G_i)$, $i, \ldots, M$ are called the Nambu brackets of second kind (NB II) [1]:

$$\frac{dF}{dt} = \sum_i \frac{\partial(F, H_i, G_i)}{\partial(x, y, z)} = \sum_i [F, H_i, G_i] \quad (3)$$

Like in [2] we call the constitutive quantities those ones which form the basis of construction of Nambu brackets but are not necessarily the constants of motion.

## Boussinesq aproximation in 3D

The Boussinesq approximation is used in the study of buoyancy-driven flows. It states that sufficiently small density $\rho$ deviations from a reference density $\rho_o$ can be neglected, except where they appear in terms which include the acceleration due to gravity. The governing equations under this aproximation are:

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla\varpi + \mathbf{b} - \mathbf{w}_a \times \mathbf{v} \quad (4)$$

$$\nabla \cdot \mathbf{v} = 0 \quad (5)$$

$$\frac{\partial b}{\partial t} = -\mathbf{v} \cdot \nabla b \quad (6)$$

Here, $\mathbf{v}$ and $b = -g(\rho - \rho_o)/\rho_o$ are the velocity and buoyancy respectively; $\mathbf{b} = b\mathbf{k}$ where $\mathbf{k}$ is the unit vector in the positive $z$ direction, oppositely to the direction of gravity acceleration of magnitude $g$ ; $-\nabla\varpi$ includes the conservative forces and the kinematic units ($\rho = 1$) are assumed; $\mathbf{w}_a = \nabla \times \mathbf{v} + 2\Omega$ is the total (absolute) vorticity in a reference frame rotating with constant angular velocity $\mathbf{\Omega}$.
We propose the following NB II's formalism:

$$\frac{d\mathcal{F}\{\mathbf{v}, b\}}{dt} = [\mathcal{F}, \mathcal{H}, \mathcal{G}]_{\mathbf{v},\mathbf{v},\mathbf{v}} + [\mathcal{F}, \mathcal{H}, \mathcal{L}]_{b,\mathbf{v},b} \quad (7)$$

$$[\mathcal{F}, \mathcal{H}, \mathcal{G}]_{\mathbf{v},\mathbf{v},\mathbf{v}} = \int d^3x \left\{ \frac{\delta\mathcal{F}}{\delta\mathbf{v}} \left( \frac{\delta\mathcal{H}}{\delta\mathbf{v}} \times \frac{\delta\mathcal{G}}{\delta\mathbf{v}} \right) \right\} \quad (8)$$

$$[\mathcal{F}, \mathcal{H}, \mathcal{L}]_{b,\mathbf{v},b} = \int d^3x \left\{ \frac{\delta\mathcal{H}}{\delta b} \nabla \left( b \frac{\delta\mathcal{F}}{\delta\mathbf{v}} \frac{\delta\mathcal{L}}{\delta b} \right) - \frac{\delta\mathcal{F}}{\delta b} \nabla \left( b \frac{\delta\mathcal{H}}{\delta\mathbf{v}} \frac{\delta\mathcal{L}}{\delta b} \right) \right\} + cyc(\mathcal{F}, \mathcal{H}, \mathcal{L}) \quad (9)$$

where we use the 3D energy $\mathcal{H}$, 3D kinetic helicity in a rotating frame, $\mathcal{G}$, [3, 4] and global bouyancy $\mathcal{L}$

$$\mathcal{H} = \int d^3x \left\{ \frac{v^2}{2} - bz \right\}$$

$$\mathcal{G} = \frac{1}{2} \int d^3x \left\{ (\nabla \times \mathbf{v} + 4\mathbf{\Omega}) \cdot \mathbf{v} \right\}$$

$$\mathcal{L} = \int d^3x b$$

## Magnetohydrodynamics (MHD)

MHD studies the electrically conducting fluids. Important examples are plasmas and liquid metals. Here we treat only the ideal case where the fluid has small resistivity and the effect of the electric field is neglected. To this approximation the governing equations are the momentum equation, the continuity equation (incompressibility is assumed) and the pre-Maxwell equations (displacement current is neglected):

$$\frac{\partial \mathbf{v}}{\partial t} = -(\nabla \times \mathbf{v}) \times \mathbf{v} - \nabla \left( \frac{p}{\rho_o} + \frac{v^2}{2} \right) + \frac{1}{\rho_o \mu} (\nabla \times \mathbf{B}) \times \mathbf{B} \quad (10)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \quad (11)$$

$$\nabla \cdot \mathbf{v} = 0 \quad (12)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (13)$$

Here $\rho_o$ is the constant density and $\mu$ the magnetic permeability (assumed constant). We construct a NB I formalism:

$$\frac{d\mathcal{F}\{\mathbf{v}, \mathbf{A}\}}{dt} = [\mathcal{F}, \mathcal{H}, \mathcal{G}]_{\mathbf{v},\mathbf{v},\mathbf{A}} \quad (14)$$

$$[\mathcal{F}, \mathcal{H}, \mathcal{G}]_{\mathbf{v},\mathbf{v},\mathbf{A}} = \int d^3x \left\{ \frac{1}{\rho_o} \frac{\delta\mathcal{F}}{\delta\mathbf{v}} \cdot \left( \frac{\delta\mathcal{H}}{\delta\mathbf{v}} \times \frac{\delta\mathcal{G}}{\delta\mathbf{A}} \right) \right\} + cyc(\mathcal{F}, \mathcal{H}, \mathcal{G}) \quad (15)$$

where $\mathbf{A}$ is the vector potential ($\mathbf{B} = \nabla \times \mathbf{A}$). We make use of the energy $\mathcal{H}$ and the cross-helicity $\mathcal{G}$ [5]:

$$\mathcal{H} = \int d^3x \left\{ \frac{\rho_o v^2}{2} + \frac{B^2}{2\mu} \right\}$$

$$\mathcal{G} = \int d^3x \mathbf{v} \cdot \mathbf{B}$$

Also, we construct a NB II formalism:

$$\frac{d\mathcal{F}\{\mathbf{v}, \mathbf{A}\}}{dt} = [\mathcal{F}, \mathcal{H}, \mathcal{K}]_{\mathbf{v},\mathbf{v},\mathbf{v}} + [\mathcal{F}, \mathcal{H}, \mathcal{L}]_{\mathbf{A},\mathbf{v},\mathbf{A}} \quad (16)$$

$$[\mathcal{F}, \mathcal{H}, \mathcal{K}]_{\mathbf{v},\mathbf{v},\mathbf{v}} = \int d^3x \left\{ \frac{1}{\rho_o} \frac{\delta\mathcal{F}}{\delta\mathbf{v}} \cdot \left( \frac{\delta\mathcal{H}}{\delta\mathbf{v}} \times \frac{\delta\mathcal{K}}{\delta\mathbf{v}} \right) \right\} \quad (17)$$

$$[\mathcal{F}, \mathcal{H}, \mathcal{L}]_{\mathbf{A},\mathbf{v},\mathbf{A}} = \int d^3x \left\{ \frac{1}{\rho_o} \frac{\delta\mathcal{F}}{\delta\mathbf{v}} \cdot \left( \frac{\delta\mathcal{H}}{\delta\mathbf{A}} \times \frac{\delta\mathcal{L}}{\delta\mathbf{A}} \right) \right\} + cyc(\mathcal{F}, \mathcal{H}, \mathcal{L}) \quad (18)$$

We use as constitutive quantities the kinetic helicity $\mathcal{K}$ and the magnetic-field helicity $\mathcal{L}$ [5]:

$$\mathcal{K} = \frac{1}{2} \int d^3x \mathbf{v} \cdot \mathbf{w}$$

$$\mathcal{L} = \int d^3x \mathbf{A} \cdot \mathbf{B}$$

Here $\mathbf{w} = \nabla \times \mathbf{v}$ is the vorticity.
A feature of the NB II is the explicit separation of the terms in the equations onto a kinetic part $[\ldots]_{\mathbf{v},\mathbf{v},\mathbf{v}}$ and a magnetic one $[\ldots]_{\mathbf{A},\mathbf{v},\mathbf{A}}$. It is clear that after taking the limit $\mathbf{B} \to \mathbf{0}$ the governing equations describe the completely incompressible fluids and also in this limit $\mathcal{L} \to 0$, whereas $\mathcal{H}$ reduces to the kinetic energy. This observation and also the fact that the NB $[\ldots]_{\mathbf{A},\mathbf{v},\mathbf{A}}$ has two functional derivatives with respect to $\mathbf{A}$ show that this magnetic bracket vanishes identicaly in this limit case. This shows directly that the kinetic helicity is conserved when the Ampere force in (10) vanishes, as should be.
So even when the kinetic helicity is not conserved the formalism of NB II provides it a non-trivial physical role.

We see that in the limit $\mathbf{B} \to \mathbf{0}$.the magnetic NB II vanishes and the helicity becomes a conserved quantity. Also we see that in this limit the NB II becomes a NB I whereas the NB I in equation (14) becomes a Poisson bracket (PB) where the corresponding PB is:

$$[\mathcal{F}, \mathcal{H}]_{\mathbf{v},\mathbf{v}} = \int d^3x \left\{ \frac{1}{\rho_o} \mathbf{w} \cdot \left( \frac{\delta\mathcal{F}}{\delta\mathbf{v}} \times \frac{\delta\mathcal{H}}{\delta\mathbf{v}} \right) \right\} \quad (19)$$

Schematically:

$$As \quad \mathbf{B} \to \mathbf{0}$$

$$[\ldots]_{\mathbf{v},\mathbf{v},\mathbf{A}} \to [\ldots]_{\mathbf{v},\mathbf{v}}$$

$$[\ldots]_{\mathbf{v},\mathbf{v},\mathbf{v}} + [\ldots]_{\mathbf{A},\mathbf{v},\mathbf{A}}, \to [\ldots]_{\mathbf{v},\mathbf{v},\mathbf{v}}$$

The same result holds for Boussinesq 2D in the limit $b \to 0$,
this suggests that there is a relation between the number of the functional derivatives in the NBs and the appearance of constitutive elements in Nambu brackets.

## Conclusions

We extended Nambu formalism onto 3D/2D Boussinesq fluids and also onto an ideal MHD case.
Two different methods for constructing Nambu brackets have been used. The first one uses the energy integral and also other existing invariants (constants of fluid motion) for such a construction. This construction was named the Nambu bracket of first kind (NB I). The second method uses the energy integral along with non-conservative quantities, like kinetic helicity (or enstrophy for Boussinesq 2D-equations), as constitutive elements for bracket construction. This is the so called Nambu brackets of second kind (NB II).

*NBII*
⇓
*NBI*
⇓
*PB*

A hierarchical relationship between NB I, NB II and a corresponding Poisson bracket (PB) is established, with regards to the limit and $\mathbf{B} \to 0$ (for Boussinesq 2D-equations $b \to 0$.
Under this limit, constitutive elements in NB II become conserved quantities and therefore NB II reduces to NB I of the simplified (limit) problem, whereas in NB I all invariants of motion except of the energy, disappear and NB I reduces to PB.

## References

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