Geometric aspects and meteorological applications of Nambu mechanics:
The motion of three point vortices in the plane

Annette Müller, Peter Névir, Institut für Meteorologie, AG Theoretische Meteorologie, Freie Universität Berlin

References

Graduate research school GeoSim who founded this work.
We thank Isabell Sonntag for providing the reanalysed data and the Helmholtz
Acknowledgements

Introduction

On synoptic scale point vortices can be seen as contracted high and
low pressure areas determined by their circulation \( \Gamma \).
\[ \Gamma > 0 : \text{cycloonic rotation} \]
\[ \Gamma < 0 : \text{anticyclonic rotation} \]

Classical representation

Helmholtz (1856)/Kirchhoff (1876)

- 6 dimensional phase space:
  \[ \mathcal{X} = \left( x_1, y_1, x_2, y_2, x_3, y_3 \right) \]
- 6 differential equations:
  \[ \begin{align*}
  \frac{dx_i}{dt} &= \frac{1}{2 \pi} \sum_{j \neq i} \Gamma_{ij} (x_j - x_i), \\
  \frac{dy_i}{dt} &= \frac{1}{2 \pi} \sum_{j \neq i} \Gamma_{ij} (y_j - y_i)
  \end{align*} \]
  with relative distances:
  \[ r_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}, \quad i,j = 1,2,3 \]

Nambu representation


- 3 dimensional phase space:
  \[ \mathcal{Y} = \left( r_1, r_2, r_3 \right) \]
- 3 differential equations:
  \[ \dot{r}_i = \frac{1}{4 \pi} \sum_{j \neq i} \Gamma_{ij} r_{ij} - \frac{1}{2} r_i, \quad \text{for } i = 1,2,3 \]

Comparison to the classical representation of three point vortices, the application
of Nambu mechanics leads to a geometrical representation
of dynamics in a three dimensional phase space. In general, point
vortex dynamics can be seen as discrete form of the 2D barotropic
and incompressible vorticity dynamic.

Summary and Outlook

Theoretically, describing the motion of three point vortices in terms of
Nambu mechanics leads to a reduction of the dimension of
the phase space. The trajectory is given by the intersection line of two surfaces
represented by two conserved quantities.

A meteorological application of three point vortices is the so-called
Omega-Blocking. The comparison of analytical results with reana-
ysed ERA-INTERIM-data show that the stationarity of blocking events
can be described by three point vortices.

It would be interesting to apply point vortices to different atmospheric
flows.

Basic Quantities:

Circulation

\[ \Gamma = \int \mathcal{X} \cdot \mathbf{v} \, d\mathbf{A} = \int \sum_{i=1}^{3} \Gamma_i \, d\mathbf{A} \]

Meridional Momentum

\[ \mathbf{P}_m = \mathbf{x} \times \mathbf{P} = \sum_{i=1}^{3} \Gamma_i (\mathbf{r_i} \times \mathbf{u}) \]

Meridional Momentum

\[ \mathbf{P}_m = \mathbf{x} \times \mathbf{P} = \sum_{i=1}^{3} \Gamma_i (\mathbf{r_i} \times \mathbf{u}) \]

Angular Momentum

\[ \Gamma = \sum_{i=1}^{3} \int \mathbf{x} \times \mathbf{u} \, d\mathbf{A} = \sum_{i=1}^{3} \Gamma_i x_i y_i \]

Zonal Momentum

\[ \mathbf{P}_z = \mathbf{y} \times \mathbf{P} = \sum_{i=1}^{3} \Gamma_i (\mathbf{r_i} \times \mathbf{u}) \]

Center of Circulation

\[ \mathbf{R}_c = \frac{\sum_{i=1}^{3} \Gamma_i \mathbf{r_i}}{\sum_{i=1}^{3} \Gamma_i} \]

Relative Angular Momentum and Total Energy

\[ \mathbf{J} = \sum_{i=1}^{3} \Gamma_i \mathbf{r_i} \times \mathbf{u} = \sum_{i=1}^{3} \Gamma_i \mathbf{r_i} \times \mathbf{u} \]

\[ \mathbf{E} = \frac{1}{2 \pi} \int \int \mathcal{X} \cdot \mathbf{v} \, d\mathbf{A} \]

Meteorological Application: Omega-Blocking

Synoptic motions can be described by geostrophic wind fields on
isentropic surfaces. Therefore, they can be considered as 2D
incompressible flows leading to an application of point vortex
theroy. As example we analyse a persistent large scale weather
situation, the so-called Omega-Blocking.

The geometric evaluated total circulation \( \Gamma_{tot} \) is given by:

\[ \Gamma_{tot} = \Gamma_1 + \Gamma_2 + \Gamma_3 = 0.03 \times 10^8 m^2/s, \quad r = 4000 km \]

The translation velocity \( \mathbf{v} \) (east to west) can be calculated by point vortex theory:

\[ \mathbf{v} = \left( \Gamma_1 + \Gamma_2 + \Gamma_3 \right) / 4 \pi r^2 = 12 m/s \]

The basic flow \( \mathbf{u} \) (west to east) was calculated by averaging of 48 time steps of reanalysed data in the mean geographic latitude
(54° N):

\[ \mathbf{u} = 15 m/s \]

Acknowledgements

We thank Isabell Sonntag for providing the reanalysed data and the Helmholtz
graduate research school GeoSim who founded this work.

References

[1] Helmholtz H. 1858: Über Integrale der hydrodynamischen Gleichungen,
Teubner
Bedeutung für die dynamische Meteorologie, Hab. Diss., FU Berlin

annette.mueller@met.fu-berlin.de